

# UNIVERSITY COLLEGE LONDON

All questions may be attempted, but only marks obtained on the first four questions will count. The use of an electronic calculator is not permitted in this examination.

## EXAMINATION FOR INTERNAL STUDENTS

1. a) (Chain rule) Let  $w(t)$  be a composite function

$$w(t) = f(X(t), Y(t))$$

MODULE CODE : **MATH1402**

ASSESSMENT : **MATH1402B**  
PATTERN

MODULE NAME : **Mathematical Methods 2**

DATE : **09-May-11**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2. a) Let  $u = (u_1, u_2, u_3)$  be a unit vector,  $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = 1$ . Let  $f(x, y, z)$  be a function of 3 variables.

Define the directional derivative  $\frac{\partial f}{\partial u}(x_0, y_0, z_0)$ .

b) Show that  $\frac{\partial f}{\partial u}(x_0, y_0, z_0)$  achieves its maximum, as a function of  $u$ ,  $|u| = 1$ , if

$$u = \frac{\nabla f(x_0, y_0, z_0)}{|\nabla f(x_0, y_0, z_0)|}$$

c) Let the surface  $S$  be given as the graph of the function  $f(x, y) = 1 + x^2 + y^2$ , where  $(x, y)$  satisfy

$$x \geq 0, y \geq 0, x + y \leq 1.$$

Find the surface integral

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**TURN OVER**

All questions may be answered, but only marks obtained on the best four questions will count. The use of an electronic calculator is not permitted in this examination.

1. a) (Chain rule) Let  $\omega(t)$  be a composite function

$$\omega(t) = f(X(t), Y(t)).$$

Write down the formula for  $\omega'(t)$  in terms of  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$  and  $X'(t)$ ,  $Y'(t)$ .

- b) Derive the formula for the chain rule of Part a).

[Hint: You may use that, near  $x = x_0, y = y_0$ ,

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0),$$

and linear Taylor's expansion of  $X(t), Y(t)$  near  $t = t_0$ .]

- c) Let  $R$  be a region on the  $xy$  plane defined by

$$x^2 + y^2 \leq 4, \quad x \geq 0, \quad y \geq 0, \quad y \leq x$$

Find the integral

$$\iint_R e^{(x^2+y^2)} y^2 dx dy.$$

2. a) Let  $\mathbf{u} = (u_1, u_2, u_3)$  be a unit vector,  $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} = 1$ . Let  $f(x, y, z)$  be a function of 3 variables.

Define the directional derivative  $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0, z_0)$ .

- b) Show that  $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0, z_0)$  achieves its maximum, as a function of  $\mathbf{u}$ ,  $|\mathbf{u}| = 1$ , if

$$\mathbf{u} = \frac{\nabla f(x_0, y_0, z_0)}{|\nabla f(x_0, y_0, z_0)|}$$

- c) Let the surface  $S$  be given as the graph of the function  $f(x, y) = 1 + x^2 + y^2$ , where  $(x, y)$  satisfy

$$x \geq 0, \quad y \geq 0, \quad x + y \leq 1.$$

Find the surface integral

$$\iint_S \frac{e^{2x+y}}{\sqrt{4z-3}} dS.$$

3. a) State the Divergence Theorem carefully.  
b) Verify the Divergence Theorem when the vector-field  $\mathbf{F}$  has the form,

$$\mathbf{F}(x, y, z) = a(xy^2\mathbf{i} + yx^2\mathbf{j}) + x^2 \cos(\pi z)\mathbf{k}, \quad a > 0,$$

and  $D$  is the cylinder,

$$x^2 + y^2 \leq 1, \quad 0 \leq z \leq 1/2.$$

4. a) State Stoke's Theorem carefully.  
b) Verify Stoke's Theorem for the vector field

$$\mathbf{F}(x, y, z) = xy\mathbf{i} - z\mathbf{j} + y^2\mathbf{k}$$

and the surface  $S$  defined by

$$z - (x^2 + y^2) = 3, \quad z \leq 4.$$

- c) Let  $R$  be a triangle on the  $xy$ -plane with vertices at the points

$$\mathbf{O} = (0, 0), \quad \mathbf{A} = (a, 0), \quad \mathbf{B} = (0, b), \quad a, b > 0.$$

Let  $\mathbf{F}$  be a vector field,

$$\mathbf{F}(x, y) = (\sin(y)e^{x \sin(y)} + y)\mathbf{i} + x \cos(y)e^{x \sin(y)}\mathbf{j}.$$

Find

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where  $C$  is the boundary of  $R$  traversed in the anti-clockwise direction.

[Hint: You may use Green's Theorem.]

5. a) Let  $D = \mathbb{R}^3 \setminus \{(x, y, z) : x = y = 0\}$ . Let  $C$  be a smooth curve in  $D$ . Let  $F(x, y, z)$  be a smooth vector field in  $D$  such that

$$\nabla \times F = 0, \quad \text{in } D.$$

Does this guarantee that  $\int_C F \cdot dr$  depends only on the initial,  $x_0 = (x_0, y_0, z_0)$ , and terminal,  $x_1 = (x_1, y_1, z_1)$ , points of  $C$ ? Explain your answer.

(Note that  $\mathbb{R}^3 \setminus \{(x, y, z) : x = y = 0\}$  is the whole  $\mathbb{R}^3$  without the  $z$ -axis).

- b) Let the vector-field  $F(x)$ ,  $x = (x, y, z) \in \mathbb{R}^3$ , be given by the formula

$$F(x, y, z) = z \cos(y) e^{xz \cos(y)} \mathbf{i} + (1 - xz \sin(y) e^{xz \cos(y)}) \mathbf{j} + x \cos(y) e^{xz \cos(y)} \mathbf{k}.$$

Is  $F$  a gradient vector-field and, if so, what is a corresponding potential?

- c) Let

$$G = z \cos(y) e^{xz \cos(y)} \mathbf{i} + (1 + xy - xz \sin(y) e^{xz \cos(y)}) \mathbf{j} + x \cos(y) e^{xz \cos(y)} \mathbf{k}.$$

Let  $C$  be a curve of the form

$$x = \cos(\pi t), \quad y = \sin(\pi t), \quad z = t(1 - t), \quad 0 \leq t \leq 1,$$

which connects the point  $x_0 = (1, 0, 0)$  with the point  $x_1 = (-1, 0, 0)$ .

Find

$$\int_C G \cdot dr.$$

[Hint: You may use Part b).]

6. a) State Fourier's Theorem carefully.  
 b) Find the Fourier coefficients of the function  $f(x)$  which is equal to
- $\pi$  for  $-\pi \leq x \leq 0$ ;
  - $\pi - x$  for  $0 \leq x < \pi$ ;
  - continued  $2\pi$  periodically from  $(-\pi, \pi)$  to the whole  $\mathbb{R}$ .
- c) Using Part b) or otherwise, show that

$$\sum_{k=1}^{\infty} \frac{8}{(2k-1)^2} = \pi^2.$$